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GRADED COALGEBRAS AND MORITA-TAKEUCHI CONTEXTS

By

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0. Introduction

Viewing a G -graded k -coalgebra over the field k as a right kG -comodule coalgebra it is possible to use a Hopf algebraic approach to the study of coalgebras graded by an arbitrary group that was started in [NT].

Let $C = \bigoplus_{g \in G} C_g$ be a G -graded coalgebra. The graded C -comodules may be viewed as comodules over the smash product $C \rtimes kG$, the general definition of which was given in [M]. Coalgebras graded by an arbitrary group have been considered in [FM] in order to introduce the notion of G -graded Hopf algebras. On the other hand, M. Takeuchi introduced in [T] the sets of pre-equivalence data connecting categories of comodules over two coalgebras (we call such a set a Morita-Takeuchi context). The main result of this note is a coalgebra version of a result established by M. Cohen, S. Montgomery in [CM] for group-graded rings: for a graded coalgebra C the coalgebras C_1 and $C \rtimes kG$ are connected by a Morita-Takeuchi context in which one of the structure maps is injective. Most of the results in this note are consequences of the foregoing. As a first application we find that a coalgebra C is strongly graded if and only if the other structure map of the context is also injective. The final section provides analogues of the Cohen-Montgomery duality theorems: if C is a coalgebra graded by the finite group G of order n , then G acts on the smash coproduct as a group of automorphisms of coalgebras and $(C \rtimes kG) \rtimes kG^*$ is coalgebra isomorphic to the comatrix coalgebra $M^c(n, C)$. If G is a finite group of order n , acting on the coalgebra D as a group of coalgebra automorphisms, then the smash coproduct $D \rtimes kG^*$ is strongly graded by G and moreover: $(D \rtimes kG^*) \rtimes kG \cong M^c(n, D)$. The second duality theorem is again a direct consequence of the Morita-Takeuchi context mentioned above.

1. Graded Coalgebras and the Smash Coproduct

Throughout this paper k is a field. We use Sweedler's "sigma" notation [S] and further notation and conventions in [T], [D]. Let G be a group with identity element 1. Recall that a k -coalgebra (C, Δ, ε) is graded by G if C is a direct sum of k -subspaces, $C = \bigoplus_{\sigma \in G} C_\sigma$, such that $\Delta(C_\sigma) \subset \sum_{xy=\sigma} C_x \otimes C_y$, for all $\sigma \in G$, and $\varepsilon(C_\sigma) = 0$ for $\sigma \neq 1$. A right C -comodule M with structure map $\rho: M \rightarrow M \otimes C$ is a graded C -comodule if $M = \bigoplus_{\sigma \in G} M_\sigma$ as k -subspaces, such that $\rho(M_\sigma) \subset \sum_{xy=\sigma} M_x \otimes C_y$ for all $\sigma \in G$. For graded right C -comodules M and N a graded comodule morphism is a C -comodule morphism $f: M \rightarrow N$ such that $f(M_\sigma) \subset N_\sigma$ for $\sigma \in G$. The category of graded right C -comodules, denoted by gr^C , is a Grothendieck category, cf. [NT]. The main purpose of this section is to develop a Hopf algebraic approach to the graded theory. First we recall, see [S] or [A], some definitions.

1.1. DEFINITION. Let H be a bialgebra over the field k , A a k -algebra and $(C, \Delta_C, \varepsilon_C)$ a k -coalgebra. Then:

- i. A is said to be a (right) H -module algebra if A is a right H -module such that $(ab) \cdot h = \sum (a \cdot h_1)(b \cdot h_2)$ and $1_A \cdot h = \varepsilon(h)1_A$ for any $h \in H$, and $a, b \in A$.
- ii. C is a right H -comodule coalgebra if C is an H -comodule by $c \mapsto \sum c_{(0)} \otimes c_{(1)}$ such that we have:

$$\sum c_{1(0)} \otimes c_{2(0)} \otimes c_{1(1)} c_{2(1)} = \sum c_{(0)1} \otimes c_{(0)2} \otimes c_{1(1)},$$

$$\sum \varepsilon_C(c_{(0)}) c_{(1)} = \varepsilon_C(c) 1_H \quad \text{for all } c \in C$$

- iii. C is a (left) H -module coalgebra if C is a left H -module such that:

$$\Delta_C(h \cdot c) = \sum h_1 \cdot c_1 \otimes h_2 \cdot c_2, \quad \varepsilon_C(h \cdot c) = \varepsilon_H(h) \varepsilon_C(c) \quad \text{for } c \in C, h \in H.$$

In the sequel we shall not refer to "right" or "left" as in the above definitions, the choice of "sides" shall remain fixed throughout.

For any group G the group algebra kG has a bialgebra structure defined by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$. The next result establishes the connection between G -graded coalgebras and kG -comodule coalgebras.

1.2. PROPOSITION. A coalgebra C graded by G can be viewed as a kG -comodule coalgebra; conversely every kG -comodule coalgebra is a G -graded coalgebra.

PROOF. For a G -graded C the map $\rho: C \rightarrow C \otimes kG$, $c \mapsto c \otimes \sigma$ for all $\sigma \in G$,

$c \in C_\sigma$, defines a kG -comodule coalgebra structure on C . Conversely, if C is a kG -comodule coalgebra then any $c \in C$ has a unique presentation $\rho(c) = \sum_{g \in G} c_g \otimes g$. Put $C_g = \{c_g, c \in C\}$, $g \in G$; C_g is a k -subspace of C . From $(I \otimes \varepsilon)\rho(c) = c \otimes 1$ we derive that $c = \sum_{g \in G} c_g$ and $C = \sum_{g \in G} C_g$. For $c \in C$, $g \in G$ we have that $c \in C_g$ if and only if $\rho(c) = c \otimes g$. If $\sum_{g \in G} c_g = 0$ for some $c_g \in C_g$ then by applying ρ we obtain $\sum c_g \otimes g = 0$ or $c_g = 0$ for all $g \in G$. Therefore $C = \bigoplus_{g \in G} C_g$. Consider $c \in C_\sigma$ and $\Delta(c) = \sum c_1 \otimes c_2$ with homogeneous c_1 's and c_2 's. From 1.1 we retain that $\sum c_1 \otimes c_2 \otimes \sigma$ equals $\sum c_1 \otimes c_2 \otimes \deg c_1 \cdot \deg c_2$, or in other words $\Delta(c)$ is the sum of all terms with $\sigma = \deg c_1 \cdot \deg c_2$, establishing that C is a G -graded coalgebra. \square

We say that the group G acts on the coalgebra D whenever there is a group morphism $\varphi: G \rightarrow \text{Aut}(D)$, the latter denoting the set of all coalgebra automorphisms of D with group structure defined as follows: if $f, g \in \text{Aut}(D)$, $f \cdot g = f \circ g$.

1.3. PROPOSITION. *If G acts on the coalgebra D then D has the structure of a kG -module coalgebra; conversely any kG -module coalgebra has a natural G -action.*

PROOF. Suppose that $\varphi: G \rightarrow \text{Aut}(D)$ determines that G acts on D then the map $kG \otimes D \rightarrow D$, $g \otimes d \mapsto \varphi(g)(d)$ defines a kG -module structure on D as desired. Conversely, if D is a kG -module coalgebra then we may define a G -action on D by $\varphi: G \rightarrow \text{Aut}(D)$, $\varphi(g)(d) = g \cdot d$ for $g \in G$, $d \in D$. \square

1.4. REMARK. Let, for a finite group G , kG^* be the dual bialgebra for the finite dimensional bialgebra kG . If the finite group G acts on the coalgebra D then D is also a kG^* -comodule coalgebra. If $\{p_g, g \in G\}$ is the dual basis of $\{g, g \in G\}$ then $\{p_g, g \in G\}$ is a system of orthogonal idempotents of kG^* . The coalgebra structure of kG^* is given in the usual way by: $\Delta(p_g) = \sum_{xy=g} p_x \otimes p_y$, $\varepsilon(p_g) = \delta_{g,1}$.

The right comodule structure of D is given by $\rho: D \rightarrow D \otimes kG^*$, $\rho(d) = \sum_{g \in G} (g \cdot d) \otimes p_g$.

In the sequel, the smash coproduct plays a central part. For a bialgebra H and an H -module coalgebra C the smash-coproduct $C \rtimes H$ is defined as the k -space $C \otimes H$ with $\Delta: C \rtimes H \rightarrow (C \rtimes H) \otimes (C \rtimes H)$ given by $\Delta(c \rtimes h) = \sum (c_1 \rtimes c_{2(1)} \cdot h_2) \otimes (c_{2(0)} \rtimes h_1)$, and $\varepsilon: C \rtimes H \rightarrow k$ given by $\varepsilon(c \rtimes h) = \varepsilon_C(c) \varepsilon_H(h)$.

1.5. PROPOSITION. $C \rtimes H$ with Δ and ε as above is a coalgebra.

PROOF. This is just the right hand version of Theorem 2.11 of [M], a proof is given in Proposition 2.3 of [FM]. \square

The smash coproduct is useful in general but has particular interest in some special cases frequently considered:

i. Graded smash coproduct

If the coalgebra C is graded by G then the coalgebra structure of $C \rtimes kG$ is given by: $\Delta(c \rtimes g) = \sum (c_1 \rtimes \deg c_2 \cdot g) \otimes (c_2 \otimes g)$, for any homogeneous $c \in C$ and $g \in G$ (where we assumed, as we will always do in the sequel, that we have used the homogeneous decomposition $\sum c_1 \otimes c_2$), whereas for all $c \in C$, $g \in G$ we have that $\varepsilon(c \rtimes g) = \varepsilon_C(c)$.

ii. If the finite group G acts on the coalgebra D , i.e. D is a kG^* -comodule coalgebra, then the coalgebra structure of $D \rtimes kG^*$ is given by:

$$\Delta(d \rtimes p_g) = \sum_{uv=g} (d_1 \rtimes p_v) \otimes (v \cdot d_2 \rtimes p_u),$$

and

$$\varepsilon(d \rtimes p_g) = \varepsilon_D(d) \delta_{g,1}, \quad \text{for all } d \in D, g \in G.$$

Note that the graded smash coproduct appears in a natural way when one studies graded comodules. Recall that a k -Abelian category is k -equivalent to a category of comodules \mathcal{M}^C over some coalgebra C if and only if it is of finite type (Theorem 5.1 of [T]). The coalgebra giving the category as a category of comodules may, in general, be a somewhat mystical object. However for a G -graded coalgebra C the k -Abelian category of graded comodules, say gr^C , is of finite type and it is therefore, equivalent to a category of comodules over the coalgebra given in the following.

THEOREM 1.6. *If C is a coalgebra graded by G then the categories gr^C and $\mathcal{M}^{C \rtimes kG}$ are isomorphic.*

PROOF. Take $M \in \text{gr}^C$ with $\rho: M \rightarrow M \otimes C$, $\rho(m) = \sum m_0 \otimes m_1$. We make M into a right $C \rtimes kG$ -comodule by defining $\rho': M \rightarrow M \otimes (C \rtimes kG)$, $m \mapsto \sum m_0 \otimes (m_1 \rtimes (\deg m)^{-1})$ for homogeneous $m \in M$. A morphism $f: M \rightarrow N$ of G -graded C -comodules is also a morphism of $C \rtimes kG$ -comodules and we have defined a functor $T: \text{gr}^C \rightarrow \mathcal{M}^{C \rtimes kG}$.

Conversely, starting from an $M \in \mathcal{M}^{C \rtimes kG}$ we obtain on M a right C -comodule

structure and a right kG -comodule structure because the linear maps $\alpha: C \rtimes kG \rightarrow C$, $c \rtimes g \mapsto c$, and $\beta: C \rtimes kG \rightarrow kG$, $c \rtimes g \mapsto \varepsilon_C(c)g^{-1}$ for $c \in C$, $g \in G$, are coalgebra morphisms. As in the proof of Proposition 1.2 it follows that $M = \bigoplus_{g \in G} M_g$ and a straightforward verification learns that M becomes a graded C -comodule. Now, for $M, N \in \mathcal{M}^{C \rtimes kG}$ and a morphism of $C \rtimes kG$ -comodules $f: M \rightarrow N$ it follows that f is also a morphism of G -graded C -comodules when M and N are viewed as such. This defines the functors $S: \mathcal{M}^{C \rtimes kG} \rightarrow \text{gr}^C$ and it is easily seen that T and S are isomorphisms of categories and inverse to each other.

1.7. REMARKS. 1. If the coalgebra C is graded by a finite group G , then the dual algebra C^* is graded by G with $C_g^* = \{f \in C^*, f(C_x) = 0 \text{ for all } x \neq g\}$. Hence C^* is a kG^* -module algebra and we may construct the smash product $C^* \# kG^*$ with multiplication given by: $(c^* \# h^*)(d^* \# g^*) = \sum (c^*(d^* \cdot h_1^*) \# g^* h_2^*)$, for all $c^*, d^* \in C^*$ and $h^*, g^* \in kG^*$. It is easy to see that the algebra $C^* \# kG^*$ is algebra-isomorphic to the dual algebra of $C \rtimes kG$.

2. If G acts on the coalgebra D via $\varphi: G \rightarrow \text{Aut}(D)$, then the group morphism $\bar{\varphi}: G \rightarrow \text{Aut}(D^*)$ given by $\bar{\varphi}(g)(d^*) = d^* \varphi(g)$ for $g \in G$, $d^* \in D^*$, defines an action of G on the algebra D^* . Note that $\text{Aut}(D^*)$ is a group with respect to $\sigma \cdot \tau = \tau \circ \sigma$ for $\sigma, \tau \in \text{Aut}(D^*)$. Thus D is a kG -module coalgebra and D^* is a kG -module algebra. If G is finite then D is a kG^* -comodule coalgebra and the dual algebra of the smash coproduct $D \rtimes kG^*$ is isomorphic to the skew group ring $D^* \# kG$.

2. The Morita-Takeuchi Context Associated to a Graded Coalgebra

The Morita-theorems for categories of comodules have been proved by M. Takeuchi in [T]; we call a set of pre-equivalence data as in [T] a Morita-Takeuchi context.

2.1. DEFINITION. A Morita-Takeuchi context $(C, D, {}_C P_D, {}_D Q_C, f, g)$ consists of coalgebras C and D , bicomodules ${}_C P_D, {}_D Q_C$ and bilinear maps $f: C \rightarrow P \square_D Q$, $g: D \rightarrow Q \square_C P$ making the following diagrams commute:

$$\begin{array}{ccc} P & \xrightarrow{\cong} & P \square_D D \\ \downarrow \cong & & \downarrow I \square g \\ C \square_C P & \xrightarrow{f \square I} & P \square_D Q \square_C P \end{array} \quad \begin{array}{ccc} Q & \xrightarrow{\cong} & Q \square_C C \\ \cong \downarrow & & \downarrow I \square f \\ D \square_D Q & \xrightarrow{g \square I} & Q \square_C P \square_D Q \end{array}$$

The context is called **strict** if f and g are injective, hence isomorphisms. In

this case the categories \mathcal{M}^C and \mathcal{M}^D of comodules over C , resp. D , are equivalent categories.

The following remark extends a corresponding one for Morita contexts given in [CRW].

2.2. PROPOSITION. *Let $(C, D, {}_C P_D, {}_D Q_C, f, g)$ be a Morita-Takeuchi context such that f is injective. Then \mathcal{M}^C is equivalent to a quotient category of \mathcal{M}^D .*

PROOF. Theorem 2.5 of [T] yields that f is an isomorphism and the exact functor $S = -\square_D Q : \mathcal{M}^D \rightarrow \mathcal{M}^C$, has a right adjoint $T = -\square_C P : \mathcal{M}^C \rightarrow \mathcal{M}^D$ such that the natural transformation $f^{-1} : ST \rightarrow Id$ is an isomorphism. By a result of P. Gabriel (cf. [G] or Proposition 15.18 of [F]) we have: $\ker S = \{X \in \mathcal{M}^D, X \square_D Q = 0\}$ is a localizing subcategory of \mathcal{M}^D and S induces an equivalence from the quotient category $\mathcal{M}/\text{Ker } S$ to \mathcal{M}^C . \square

2.3. COROLLARY. *Let $(C, D, {}_C P_D, {}_D Q_C, f, g)$ be a Morita-Takeuchi context such that f is injective then g is injective (i.e. the context is strict) if and only if ${}_D Q$ is faithfully coflat.* \square

PROOF. By Proposition 2.2 the injectivity of g is equivalent to S being an equivalence, again equivalent to $\text{Ker } S = \{0\}$ or ${}_D Q$ being faithfully coflat. \square

Before establishing the main result of this section let us point out that there is a natural way to associate a graded coalgebra to a given Morita-Takeuchi context. Indeed, if we have a Morita-Takeuchi context $(C, D, {}_C P_D, {}_D Q_C, f, g)$ let $x \mapsto \sum x_{-1} \otimes x_0$, resp. $x \mapsto \sum x_{(0)} \otimes x_{(1)}$, be the left, resp. right, comodule structure of P , resp. Q . The image of $u \in C$ (resp. D) under f (resp. g) in $P \square_D Q$ (resp. $Q \square_C P$) will be denoted by $\sum f(u)_1 \otimes f(u)_2$. (resp. $\sum g(u)_1 \otimes g(u)_2$).

Put $\Gamma = \begin{pmatrix} C & P \\ Q & D \end{pmatrix} = \left\{ \begin{pmatrix} c & p \\ q & d \end{pmatrix}, c \in C, d \in D, p \in P, q \in Q \right\}$.

We make Γ into a coalgebra by defining $\Delta : \Gamma \rightarrow \Gamma \otimes \Gamma$ as follows:

$$\begin{aligned} \Delta \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} &= \sum \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} + \sum \begin{pmatrix} 0 & f(c)_1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ f(c)_2 & 0 \end{pmatrix} \\ \Delta \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} &= \sum \begin{pmatrix} 0 & 0 \\ 0 & d_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ d_2 & 0 \end{pmatrix} + \sum \begin{pmatrix} 0 & 0 \\ g(d)_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & g(d)_2 \\ 0 & 0 \end{pmatrix} \\ \Delta \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} &= \sum \begin{pmatrix} p_{-1} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & p_0 \\ 0 & 0 \end{pmatrix} + \sum \begin{pmatrix} 0 & p_{(0)} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & p_{(1)} \end{pmatrix} \\ \Delta \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} &= \sum \begin{pmatrix} 0 & 0 \\ 0 & q_{-1} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ q_0 & 0 \end{pmatrix} + \sum \begin{pmatrix} 0 & 0 \\ q_{(0)} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & q_{(1)} \end{pmatrix} \end{aligned}$$

for $c \in C$, $d \in D$, $p \in P$, $q \in Q$, and extended linearly, $\varepsilon: \Gamma \rightarrow k$ given by $\varepsilon \begin{pmatrix} c & p \\ q & d \end{pmatrix} = \varepsilon_C(c) + \varepsilon_D(d)$. Moreover Γ is \mathbb{Z} -graded by putting $\Gamma_0 = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$, $\Gamma_{-1} = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$ and $\Gamma_1 = \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}$, $\Gamma_k = 0$ for $k \neq -1, 0, 1$.

Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a coalgebra, graded by G . Recall from [NT] that C_1 is a coalgebra with comultiplication $\Delta_1: C_1 \rightarrow C_1 \otimes C_1$ given by $\Delta_1(c) = \sum \pi(c_1) \otimes \pi(c_2) = \sum \pi(c_1) \otimes c_2 = \sum c_1 \otimes \pi(c_2)$ for all $c \in C_1$, where $\pi: C \rightarrow C_1$ is the natural projection. The co-unit of C_1 is just ε_C restricted to C_1 . Since π is a coalgebra map, C becomes a left C_1 -comodule via the structure map $\rho_1^l: C \rightarrow C_1 \otimes C$, $c \mapsto \sum \pi(c_1) \otimes c_2$ (c homogeneous) and it becomes a right C_1 -comodule via $\rho_1^r: C \rightarrow C \otimes C_1$, $c \mapsto \sum c_1 \otimes \pi(c_2)$ (c homogeneous). Now C is a graded right C -comodule, so by Theorem 1.6 C is a right $C \rtimes kG$ -comodule via the map

$$\rho_2^r: C \rightarrow C \otimes (C \rtimes kG), \quad c \mapsto \sum c_1 \otimes (c_2 \rtimes (\deg c)^{-1})$$

for c homogeneous. For any homogeneous $c \in C$, we have $(I \otimes \rho_2^r) \rho_1^l(c) = (\rho_1^l \otimes I) \rho_2^r(c) = \sum \pi(c_1) \otimes c_2 \otimes (c_3 \rtimes (\deg c)^{-1})$; thus C becomes a left C_1 , right $C \rtimes kG$ -bicomodule. In a similar way C becomes a left $C \rtimes kG$, right C_1 -bicomodule where the left $C \rtimes kG$ -comodule-structure of C is given by $\rho_2^l(c) = \sum (c_1 \rtimes \deg c_2) \otimes c_3$, for any homogeneous $c \in C$.

Define $f: C_1 \rightarrow C \square_{C \rtimes kG} C$, $c \mapsto \sum c_1 \otimes c_2 = \Delta_C(c)$. Observe that for any $c \in C_1$ we obtain:

$$\begin{aligned} \sum \rho_2^r(c_1) \otimes c_2 &= \sum c_1 \otimes c_2 (\deg c_2)^{-1} (\deg c_1)^{-1} \otimes c_3 \\ &= \sum c_1 \otimes c_2 \otimes \deg c_3 \otimes c_3 \\ &= \sum c_1 \otimes \rho_2^l(c_2) \end{aligned}$$

so the definition of f above is satisfactory. Moreover, f is a morphism of left and right C_1 -comodules as is easily verified. Note also that f is injective because it is the restriction of the comultiplication of C to C_1 .

Next define $g: C \rtimes kG \rightarrow C \square_{C_1} C$, $c \rtimes x \mapsto \sum c_1 \otimes \pi_{x^{-1}}(c_2)$ for $x \in G$ and homogeneous $c \in C$, where π_x denotes the projection from C to C_x . In order to have that g is well-defined it is necessary that: $\sum (c_1)_1 \otimes \pi((c_1)_2) \otimes \pi_{x^{-1}}(c_2) = \sum c_1 \otimes \pi(\pi_{x^{-1}}((c_2)_1)) \otimes \pi_{x^{-1}}((c_2)_2)$. However the left hand side is obtained from $\sum c_1 \otimes c_2 \otimes c_3$ by collecting the terms with $\deg c_2 = 1$ and $\deg c_3 = x^{-1}$; on the other hand the right hand sum is an expression of the same thing. Moreover g is a morphism of right (and left) $C \rtimes kG$ -comodules; this follows from: $\sum_{\deg c_2 = x^{-1}} (c_1 \otimes$

$(c_2)_1) \otimes ((c_2)_2 \rtimes x) = \sum_{\deg(c_1)_2 = x^{-1}(\deg c_2) - 1} ((c_1)_1 \otimes (c_1)_2) \otimes (c_2 \rtimes x)$ because both members are actually equal to: $\sum_{\deg c_2 \deg c_3 = x^{-1}} (c_1 \otimes c_2) \otimes (c_3 \rtimes x)$. The other assertion (left) follows in a similar way.

2.4. THEOREM. *With notation as above: $(C_1, C \rtimes kG, {}_{C_1}C_{C \rtimes kG, C \rtimes kG}C_{C_1}, f, g)$ is a Morita-Takeuchi context. The map f is injective hence an isomorphism.*

PROOF. The only thing left to be proved is that f and g do satisfy the compatibility conditions, i.e. the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{\theta} & C \square_{C \rtimes kG} C \rtimes kG \\ \cong \downarrow \psi & & \downarrow I \square g \\ C_1 \square_{C_1} C & \xrightarrow{f \square I} & C \square_{C \rtimes kG} C \square_{C_1} C \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\theta'} & C \square_{C_1} C \\ \cong \downarrow \psi' & & \downarrow I \square f \\ (C \rtimes kG) \square_{C \rtimes kG} C & \xrightarrow{g \square I} & C \square_{C_1} C \square_{C \rtimes kG} C. \end{array}$$

Now for $c \in C_x$ we have: $(I \square g)\theta(c) = (I \square g)(\sum c_1 \otimes (c_2 \rtimes x^{-1})) = \sum c_1 \otimes c_2 \otimes \pi_x(c_3) = \sum_{\deg c_3 = x} c_1 \otimes c_2 \otimes c_3$, and also $(f \square I)(\psi(c)) = (f \square I)(\sum \pi(c_1) \otimes c_2) = (f \square I)(\sum_{\deg c_1 = 1} c_1 \otimes c_2) = (f \square I)(\sum_{\deg c_2 = x} c_1 \otimes c_2) = \sum_{\deg c_3 = x} c_1 \otimes c_2 \otimes c_3$.

That proves commutativity of the first diagram. For the second diagram we just compute: $(I \square f)\theta'(c) = (I \square f)(\sum c_1 \otimes \pi(c_2)) = (I \square f)(\sum_{\deg c_2 = 1} c_1 \otimes c_2) = (I \square f)(\sum_{\deg c_1 = x} c_1 \otimes c_2) = \sum_{\deg c_1 = x} c_1 \otimes c_2 \otimes c_3$ and also $(g \square I)\psi'(c) = (g \square I)(\sum (c_1 \rtimes \deg c_2) \otimes c_2) = \sum_{\deg c_2 = (\deg c_3) - 1} c_1 \otimes c_2 \otimes c_3 = \sum_{\deg c_2 \deg c_3 = 1} c_1 \otimes c_2 \otimes c_3 = \sum_{\deg c_1 = x} c_1 \otimes c_2 \otimes c_3$. \square

2.5. COROLLARY. *If $C = \bigoplus_{\sigma \in G} C_\sigma$ is a graded coalgebra then \mathcal{M}^{C_1} is equivalent to a quotient category of gr^C .*

PROOF. A consequence of Theorem 1.6, Theorem 2.4 and Proposition 2.2. \square

Recall that a G -graded coalgebra $C = \bigoplus_{\sigma \in G} C_\sigma$ is said to be strongly graded if the canonical k -linear map $\gamma_{u,v}: C_{u,v} \rightarrow C_u \otimes C_v$, $c \mapsto \sum \pi_u(c_1) \otimes \pi_v(c_2)$, is injective for all $u, v \in G$ (see [NT]). The next result establishes that strongly graded coalgebras may be characterized using the Morita-Takeuchi context from Theorem 2.4 just like in the case of group-graded rings (see [CM]).

2.6. COROLLARY. *Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a G -graded coalgebra, then the following assertions are equivalent:*

1. C is strongly G -graded
2. The context given in Theorem 2.4. is strict
3. C is faithfully coflat as a left $C \rtimes kG$ -comodule.

PROOF. $2.\Rightarrow 1.$ Take $u, v \in G$ and $c \in C_{uv}$ such that we have: $\gamma_{u,v}(c) = \sum \pi_u(c_1) \otimes \pi_v(c_2) = 0$. Then $g(c \rtimes v^{-1}) = \sum c_1 \otimes \pi_v(c_2) = \sum \pi_u(c_1) \otimes \pi_u(c_2) = 0$, hence $c \rtimes v^{-1} = 0$ and $c = 0$.

$1.\Rightarrow 2.$ Let $\alpha = \sum c_i \rtimes x_i \in C \rtimes kG$ with c_i homogeneous of degree σ_i . Suppose that for $i \neq j$ we have $(\sigma_i, x_i) \neq (\sigma_j, x_j)$. If $g(\alpha) = 0$ then $\sum_{i, (c_i)} (c_i)_1 \otimes \pi_{x_i^{-1}}((c_i)_2) = 0$, therefore $\sum_{i, (c_i)} \pi_{\sigma_i x_i}((c_i)_1) \otimes \pi_{x_i^{-1}}((c_i)_2) = 0$. On the other hand: $\pi_{\sigma_i x_i}((c_i)_1) \otimes \pi_{x_i^{-1}}((c_i)_2) \in C_{\sigma_i x_i} \otimes C_{x_i^{-1}}$. Since $C \otimes C = \bigoplus_{u, v \in G} C_u \otimes C_v$ we obtain for fixed i , the relation: $\sum_{(c_i)} \pi_{\sigma_i x_i}((c_i)_1) \otimes \pi_{x_i^{-1}}((c_i)_2) = 0$. The latter yields $\gamma_{\sigma_i x_i, x_i^{-1}}(c_i) = 0$ and therefore $c_i = 0$ for every choice of i , i.e. $\alpha = 0$ follows.

$2.\Leftrightarrow 3.$ Follows from Corollary 2.3. □

As a further application we reobtain Theorem 5.3 of [NT] which is a co-algebra version of a well-known result of E. Dade.

2.7. COROLLARY. *The graded coalgebra C is strongly graded if and only if the induced functor $-\square_{C_1} C: \mathcal{M}^{C_1} \rightarrow \text{gr}^C$ is an equivalence of categories.*

2.8. REMARK. The functor $(-)_1: \text{gr}^C \rightarrow \mathcal{M}^{C_1}$, $M \mapsto M_1$, is naturally isomorphic to the functor $-\square_{C \rtimes kG} G$ since they are both left adjoints of the induced functor $-\square_{C_1} C$ (see [NT] Proposition 4.1, [T] Remark 2.4). Therefore the localizing category implicit in Corollary 2.5 is just $\text{Ker}(-)_1 = \text{Ker}(-\square_{C \rtimes kG} C)$.

As a final application of these techniques let us include a short proof of Corollary 6.4 in [NT].

2.9. COROLLARY. *If C is a strongly graded coalgebra for the group G then G is a finite group.*

PROOF. If G is infinite we could select a non-zero homogeneous $c \in C$ and $x \in G$ such that $x \neq \deg(c_2)^{-1}$ for all c_2 . Then $g(c \rtimes x) = 0$, but that would contradict injectivity of g . □

3. Duality.

For a quasi-finite right C -comodule M , the so-called coalgebra of “co-endomorphisms” of M has been defined in [T., 1.17] and it is denoted by $e_{-C}(M)$. Unfortunately this coalgebra is not easy to use because of the rather complex comultiplication, so it will be useful to give a nicer description of $e_{-C}(M)$ in some particular situation, e.g. in case M is a finitely cogenerated free-comodule (that is, $M \cong X \otimes C$, for some finite dimensional k -vectorspace X , with the obvious

comodule structure).

Let C be a coalgebra, X an n -dimensional k -space with basis $\{x_1, \dots, x_n\}$. Consider the $n \times n$ comatrix coalgebra $M^c(n, k)$ which is a k -space with basis $\{x_{ij}, 1 \leq i, j \leq n\}$ and Δ, ε given as follows: $\Delta(x_{ij}) = \sum_p x_{ip} \otimes x_{pj}$, $\varepsilon(x_{ij}) = \delta_{ij}$.

The $n \times n$ comatrix coalgebras over C , denoted by $M^c(n, C)$ is defined to be the tensor product of coalgebra $C \otimes M^c(n, k)$. We endow $C \otimes X$ with a left C - and a right $M^c(n, C)$ -bicomodule structure as follows. The left C -comodule structure is given by the map: $\rho_1^l: C \otimes X \rightarrow C \otimes C \otimes X$, $c \otimes x \mapsto \sum c_1 \otimes c_2 \otimes x$. The right $M^c(n, C)$ -comodule structure is given by the map: $\rho_2^r: C \otimes X \rightarrow C \otimes X \otimes M^c(n, C)$, $c \otimes x_i \mapsto \sum_p c_1 \otimes x_p \otimes c_2 \otimes x_{pi}$.

In a similar way $C \otimes X$ is a left $M^c(n, C)$ -right C -bicomodule via the structure maps:

$$\begin{aligned} \rho_1^r: C \otimes X &\rightarrow C \otimes X \otimes C, \quad c \otimes x \mapsto \sum c_1 \otimes x \otimes c_2 \\ \rho_2^l: C \otimes X &\rightarrow M^c(n, C) \otimes C \otimes X, \quad c \otimes x_i \mapsto \sum_p c_1 \otimes x_{ip} \otimes c_2 \otimes x_p \end{aligned}$$

Define $f: C \rightarrow (C \otimes X) \square_{M^c(n, C)} (C \otimes X)$, $c \mapsto \sum_{i, (c)} (c_i \otimes x_i) \otimes (c_2 \otimes x_i)$, which is obviously injective and C -bilinear. Define $g: M^c(n, C) \rightarrow (C \otimes X) \square_C (C \otimes X)$, $c \otimes x_{ij} \mapsto \sum (c_1 \otimes x_i) \otimes (c_2 \otimes x_j)$ which is also injective and $M^c(n, C)$ -bilinear. One easily verifies the following relations:

$$\begin{aligned} (I \square f) \rho_1^r(c \otimes x_i) &= (g \square I) \rho_2^l(c \otimes x_i) = \sum_p c_1 \otimes x_i \otimes c_2 \otimes x_p \otimes c_3 \otimes x_p \\ (f \square I) \rho_1^l(c \otimes x_i) &= (I \square g) \rho_2^r(c \otimes x_i) = \sum_p c_1 \otimes x_p \otimes c_2 \otimes x_p \otimes c_3 \otimes x_i \end{aligned}$$

According to results of [T] we immediately obtain:

3.1. PROPOSITION. $(C, M^c(n, C), C \otimes X, C \otimes X, f, g)$ is a strict Morita-Takeuchi context. In particular we have coalgebra isomorphisms:

$$e_{C-}(C \otimes X) \cong M^c(n, C) \cong e_{-C}(C \otimes X)$$

3.2. THEOREM. Let G be a finite group acting on the coalgebra D , then $D \rtimes kG^*$ is a strongly graded coalgebra and there exist coalgebra isomorphisms:

$$(D \rtimes kG^*) \rtimes kG \cong e_{D-}(D \rtimes kG^*) \cong M^c(n, D)$$

where $n = |G|$.

PROOF. The map $\rho: D \rtimes kG^*$, $d \mapsto \sum_g (g \cdot d) \otimes p_g$, makes D into a kG^* -comodule. The comultiplication of $D \rtimes kG^*$ is given by $\Delta(d \rtimes p_x) = \sum_{uv=x} (d \rtimes p_v) \otimes (v d_2 \rtimes p_u)$. This establishes that $D \rtimes kG^*$ is a graded coalgebra of type G with grading given by $(D \rtimes kG^*)_g = D \rtimes p_{g^{-1}}$. The canonical morphism $D \rtimes p_1 \rightarrow$

$(D \rtimes p_{\sigma^{-1}}) \otimes (D \rtimes p_{\sigma})$, $d \rtimes p_1 \mapsto \sum (d_1 \rtimes p_{\sigma^{-1}1}) \otimes (\sigma^{-1}d_2 \rtimes p_{\sigma})$, is clearly injective. Thus $D \rtimes kG^*$ is a strongly graded coalgebra, and $(D \rtimes kG^*)_1 = D \rtimes p_1 \cong D$. Applying the Morita-Takeuchi context (constructed in Section 2) to $D \rtimes kG^*$, we have a strict context and so it provides us with coalgebra isomorphisms:

$$(D \rtimes kG^*) \rtimes kG \cong e_{(D \rtimes p_1)} (D \rtimes kG^*) \cong e_{D^-} (D \rtimes kG^*).$$

The left $(D \rtimes p_1)$ -structure of $D \rtimes kG^*$ is given by $d \rtimes p_x \mapsto \sum (d_1 \rtimes p_1) \otimes (d_2 \rtimes p_x)$, and this yields exactly the left D -comodule structure of $D \otimes X$ where $X = kG^*$ is a k -space of dimension n . Proposition 3.1 yields the second isomorphism. \square

A similar result holds for graded coalgebras (or coactions).

3.3. THEOREM. *Let C be a coalgebra graded by the finite group G . Then G acts on the coalgebra $C \rtimes kG$ and there are coalgebra isomorphisms:*

$$(C \rtimes kG) \rtimes kG^* \cong e_{C^-} (C \rtimes kG) \cong M^c(n, C)$$

PROOF. An action of G on the coalgebra $C \rtimes kG$ is given by $h \cdot (c \rtimes g) = c \rtimes gh^{-1}$, $g, h \in G$ and $c \in C$. Thus $C \rtimes kG$ becomes a kG^* -comodule coalgebra via the map:

$$c \rtimes g \mapsto \sum_y y \cdot (c \rtimes g) \otimes p_y = \sum_y (c \rtimes gy^{-1}) \otimes p_y.$$

The comultiplication of $(C \rtimes kG) \rtimes kG^*$ is given by

$$\Delta((c \rtimes x) \rtimes p_g) = \sum_{uv=g} ((c_1 \rtimes \deg c_2 \cdot x) \rtimes p_v) \otimes ((c_2 \rtimes xv^{-1}) \rtimes p_u)$$

for any $x, g \in G$ and homogeneous $c \in C$. Now let $\{e_{x,y}, x, y \in G\}$ be a basis for $M^c(n, k)$. Define a map $F: (C \rtimes kG) \rtimes kG^* \rightarrow M^c(n, C)$, $(c \rtimes x) \rtimes p_g \mapsto c \otimes e_{\alpha, \beta}$ where $\alpha = \deg c \cdot x$, $\beta = xg^{-1}$ for $x, g \in G$ and homogeneous $c \in C$. Let us check that F is a coalgebra morphism. Indeed,

$$\begin{aligned} \Delta(F((c \rtimes x) \rtimes p_g)) &= \Delta(c \otimes e_{\alpha, \beta}) \\ &= \sum_{z, (c)} (c_1 \otimes e_{\alpha, z}) \otimes (c_2 \otimes e_{z, \beta}) \end{aligned}$$

and also

$$\begin{aligned} (F \otimes F)(\Delta((c \rtimes x) \rtimes p_g)) &= \sum_{uv=g} (c_1 \otimes e_{\deg c_1 \deg c_2 x \deg c_2 xv^{-1}}) \otimes (c_2 \otimes e_{\deg c_2 xv^{-1}, xv^{-1}, u^{-1}}) \\ &= \sum_v (c_1 \otimes e_{\alpha, \deg c_2 xv^{-1}}) \otimes (c_2 \otimes e_{\deg c_2 xv^{-1}, \beta}). \end{aligned}$$

Since $\{\deg c_2 xv^{-1}, v \in G\} = G$, both sums are equal. Now, consider $(c \rtimes x) \rtimes p_g \in (C \rtimes kG) \rtimes kG^*$ for $x, g \in G$ and c homogeneous. Write ε for the co-unit of $(C \rtimes kG) \rtimes kG^*$ and ε' for the co-unit of $M^c(n, C)$. Then we have:

$$\begin{aligned}
\varepsilon((c \rtimes x) \rtimes p_g) &= \varepsilon_C(c) \delta_{\deg c, 1} \delta_{g, 1} \\
\varepsilon'(c \otimes e_{\alpha, \beta}) &= \varepsilon_C(c) \delta_{\deg c, 1} \delta_{\deg c x, x g^{-1}} \\
&= \varepsilon_C(c) \delta_{\deg c, 1} \delta_{x, x g^{-1}} = \varepsilon_C(c) \delta_{\deg c, 1} \delta_{1, g^{-1}} \\
&= \varepsilon_C(c) \delta_{\deg c, 1} \delta_{g, 1}.
\end{aligned}$$

Therefore F is a coalgebra map as claimed. Now define $H: M^e(n, C) \rightarrow (C \rtimes kG) \rtimes kG^*$ by putting $H(c \otimes e_{u, v}) = (c \rtimes (\deg c)^{-1}u) \rtimes p_{v^{-1}(\deg c)^{-1}u}$, for $u, v \in G$ and homogeneous $c \in C$. Again H is a coalgebra morphism because:

$$\begin{aligned}
\Delta(H(c \otimes e_{u, v})) &= \sum_{zt=v^{-1}(\deg c)^{-1}u} ((c_1 \rtimes \deg c_2(\deg c)^{-1}u) \rtimes p_t) \otimes ((c_2 \rtimes (\deg c)^{-1}u t^{-1}) \rtimes p_z) \\
(H \otimes H)(\Delta(c \otimes e_{u, v})) &= (H \otimes H)(\sum_h (c_1 \otimes e_{u, h}) \otimes (c_2 \otimes e_{h, v})) \\
&= \sum_h ((c_1 \rtimes (\deg c_1)^{-1}u) \rtimes p_{h^{-1}(\deg c_1)^{-1}u}) \otimes ((c_2 \rtimes (\deg c_2)^{-1}h) \rtimes p_{v^{-1}(\deg c_2)^{-1}h}).
\end{aligned}$$

For fixed c_1 and u we have that $\{h^{-1}(\deg c_1)^{-1}u, h \in G\} = G$ and if we write $t = h^{-1}(\deg c_1)^{-1}u$, $z = v^{-1}(\deg c_2)^{-1}h$, then the above sums are clearly equal as desired. The fact that H preserves the co-unit too is obvious. Finally it is clear that $F \cdot H$ and $H \cdot F$ are the identities so that we do arrive at a coalgebra isomorphism. The isomorphism involving $e_{c-}(C \rtimes kG)$ is obvious because of Proposition 3.1 (the left C -comodule structure of $C \rtimes kG$ is given by $c \rtimes g \mapsto \sum c_1 \otimes (c_2 \rtimes g)$). \square

3.4. COROLLARY. *There exists a strict Morita-Tekeuchi context connecting C and $(C \rtimes kG) \rtimes kG^*$.*

PROOF. $C \rtimes kG$ is a left C -comodule that is a quasi-finite injective co-generator (in view of Proposition 3.1 and [T]). Moreover $C \rtimes kG$ is a right $(C \rtimes kG) \rtimes kG^*$ -comodule via $c \rtimes g \mapsto \sum_u (c_1 \rtimes \deg c_2 g u) \otimes (c_2 \rtimes g u) \rtimes p_{u^{-1}}$, for $g \in G$ and homogeneous $c \in C$. Hence $C \rtimes kG$ is a C – $(C \rtimes kG) \rtimes kG^*$ -bicomodule. The assertion now follows from [T, Theorem 3.5 iv]. \square

3.5. REMARKS. The Morita-Takeuchi context of the above corollary may be given in detail. This may have an independent interest because it provides another proof of Theorem 3.3 and provides a hint for establishing a more general duality result we do not dwell upon here. The second bicomodule is also $C \rtimes kG$ with right C -comodule structure given by the map: $c \rtimes g \mapsto \sum (c_1 \rtimes \deg c_2 g) \otimes c_2$ (for homogeneous c) and left $(C \rtimes kG) \rtimes kG^*$ -comodule struc-

ture given by: $c \rtimes g \rightarrow \sum_h (c_1 \rtimes \deg c_2 g) \rtimes p_h \otimes (c_2 \rtimes gh)$ (for homogeneous c) we have $f: C \rightarrow (C \rtimes kG) \square_{(C \rtimes kG) \rtimes kG^*} (C \rtimes kG)$, $f(c) = \sum_h (c_1 \rtimes \deg c_2 h) \otimes (c_2 \rtimes h_2)$ for homogeneous $c \in C$, $g: (C \rtimes kG) \rtimes kG^* \rightarrow (C \rtimes kG) \square_c (C \rtimes kG)$, $g((c \rtimes g) \rtimes p_h) = \sum (c_1 \rtimes \deg c_2 g) \otimes (c_2 \rtimes gh)$, for homogeneous $c \in C$. It is also easily seen that f and g are injective maps.

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Added in proof. A general duality result for crossed coproducts was proved by S. Dăscălescu, S. Raianu, Y. Zhang in “Finite Hopf-Galois coextensions, crossed coproducts and duality”, to appear in J. Algelma.